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THE MASS POWER SPECTRUM IN QUINTESSENCE COSMOLOGICAL MODELS

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ABSTRACT

We present simple analytic approximations for the linear and fully evolved nonlinear mass power spectrum of matter density fluctuations for spatially flat cold dark matter (CDM) cosmological models with quintessence (Q). Quintessence is a time-evolving, spatially inhomogeneous energy component with negative pressure and an equation of state $w_Q < 0$. It clusters gravitationally on large length scales but remains smooth like the cosmological constant on small length scales. We show that the clustering scale is determined by the Compton wavelength of the Q-field and derive a shape parameter, Γ_Q, to characterize the linear mass power spectrum. The growth of linear perturbations as functions of redshift, w_Q, and matter density, Ω_m, is also quantified. Calibrating to *N*-body simulations, we construct a simple extension of Ma's 1998 formula that closely approximates the nonlinear power spectrum for a range of plausible QCDM models.

Subject headings: cosmology: theory — dark matter — large-scale structure of universe — methods: analytical

1. INTRODUCTION

Quintessence (Q) offers an alternative to the cosmological constant (Λ) as the missing energy in a spatially flat universe with a subcritical matter density Ω_m (Caldwell, Dave, & Steinhardt 1998 and references therein). It is an energy component which, similar to Λ, has negative pressure and therefore a negative w_Q in the equation of state $p_Q = w_Q \rho_Q$. However, unlike Λ, for which $w = -1$, quintessence is time evolving and spatially inhomogeneous, and w_Q can have a range of values. The observational imprints of the quintessence therefore differ from those of the commonly studied cold dark matter with a nonzero Λ (ΛCDM) cosmology (e.g., Wang et al. 1999).

In this Letter, we study spatially flat QCDM models in which the cold dark matter and Q-field together make up the critical density (i.e., $\Omega_m + \Omega_Q = 1$). The quintessence is modeled as a scalar field that evolves with a constant equation of state w_Q. It drives the cosmological expansion at late times, influencing the rate of growth of structure. Fluctuations in Q behave as an ultralight mass scalar field: on very large length scales the quintessence clusters gravitationally, thereby modifying the level of cosmic microwave background temperature anisotropy relative to the matter power spectrum amplitude (in addition to a late-time–integrated Sachs-Wolfe effect); on small length scales, fluctuations in Q disperse relativistically and the Q-field behaves as a smooth component.

We investigate the effects of the quintessence on the spectrum and time evolution of gravitational clustering in both the linear and nonlinear regimes. We propose simple, analytical fitting formulas for both the linear and fully evolved nonlinear power spectrum of matter density fluctuations in plausible QCDM models. For the linear power spectrum (§ 2), we introduce a simple parameter Γ_Q derived from the Compton wavelength of the Q-field to characterize its shape. This parameter determines the length scale above which the Q-field can cluster gravitationally and is reminiscent of Γ_ν derived from the free-streaming distance of hot neutrinos in cold+hot dark matter (C+HDM) models by Ma (1996). For the nonlinear power spectrum (§ 3), we examine the validity of the simple linear-to-nonlinear mapping technique that has been successfully developed for scale-free, CDM, C+HDM, and ΛCDM models (Hamilton et al. 1991; Jain, Mo, & White 1995; Peacock & Dodds 1996; Ma 1998). We present a simple extension of the analytical formula of Ma (1998) that closely approximates the QCDM nonlinear power spectrum computed from a set of *N*-body simulations.

The formulae presented in this Letter are essential for gaining physical insight into the effects of the quintessence on gravitational collapse and for performing rapid predictions of observable quantities in the linear as well as nonlinear regimes in plausible QCDM models.

2. LINEAR POWER SPECTRUM

We use the conventional form to express the linear power spectrum[5] for the matter density perturbation δ_m in QCDM models:

$$P(k,\, a) = A_Q k^n T_Q^2(k) \left(\frac{a g_Q}{g_{Q,0}}\right)^2 , \qquad (1)$$

where A_Q is a normalization, k is the wavenumber, n is the spectral index of the primordial adiabatic density perturbations, and T_Q is the transfer function which encapsulates modifications to the primordial power-law spectrum. The function g_Q is the linear growth suppression factor, $g_Q = D/a$, where D is the standard linear growth factor for the matter density field in QCDM models and $g_{Q,0} = g_Q(a = 1)$ denotes its value at the present day with scale factor $a = 1$. We discuss each piece of equation (1) in turn.

First we examine the transfer function for the matter density field. To isolate the effects of quintessence, we find it convenient and illuminating to compare a pair of QCDM and ΛCDM models that have the same set of cosmological parameters and differ only in w_Q (recall $w = -1$ for ΛCDM). We define the relative transfer function for a pair of such models to be $T_{Q\Lambda} = T_Q/T_\Lambda$. For T_Λ, we follow the convention and set the arbitrary amplitude of T_Λ to unity as $k \to 0$. The form of T_Λ is well known, and various fitting formulae have been published

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[5] Defined so that the two-point correlation function is $\xi(r) \equiv 4\pi \int k^2 dk P(k) \sin(kr)/(kr)$.

(e.g., Bardeen et al. 1986; Efstathiou, Bond, & White 1992; Sugiyama 1995). More complicated fits with higher accuracy have also been developed for higher baryon ratios ($\Omega_b/\Omega_m \gtrsim 20\%$) and for the features due to baryonic oscillations and damping (e.g., Bunn & White 1997; Eisenstein & Hu 1998).

The transfer function T_Q for QCDM models resembles T_Λ for ΛCDM, but with one key difference. The linear matter density field, δ_m, evolves according to the equation $\ddot{\delta}_m + 2H\dot{\delta}_m = 4\pi G(\rho_m\delta_m + \delta\rho_Q + 3\delta p_Q)$, where $H = \dot{a}/a$ and the dots denote differentiation with respect to proper time. On small length scales, the Q-field is smooth ($\delta\rho_Q,\ \delta p_Q \ll \delta\rho_m$), and we recover the familiar equation for the evolution of δ_m (Caldwell et al. 1998). On very large length scales, however, the Q-field clusters and contributes to the energy density and pressure perturbations. The result is a different growth rate for δ_m on large and small scales once the quintessence starts to dominate the cosmological energy density. We can determine the characteristic scale separating these two regimes by examining the linear equation for the Q-field: $\ddot{\delta Q} + 3H\dot{\delta Q} + (k^2 + V_{,QQ})\delta Q = \dot{\delta}_m[(1 + w_Q)\rho_Q]^{1/2}$ (where V is the Q-field potential, $V_{,QQ} \equiv d^2V/dQ^2$, and $\rho_Q = \dot{Q}^2/2 + V$). We see that δQ itself behaves as a scalar field with an effective mass $(V_{,QQ})^{1/2}$ and a Compton wavenumber of $k_Q \sim (V_{,QQ})^{1/2}$. On small length scales (i.e., $k \gg k_Q$), the amplitude of δQ and hence $\delta\rho_Q$ is damped and does not enter the evolution equation for δ_m. On large scales ($k \ll k_Q$) δQ grows, so the Q-field clusters and in turn affects the evolution of δ_m.

The change in the behavior of δ_m near $k \sim (V_{,QQ})^{1/2}$ as a result of differing Q-clustering properties is illustrated in Figure 1 for $T_{Q\Lambda}$ versus k for a range of w_Q and Ω_m. We have chosen to normalize $T_{Q\Lambda}$ to unity at the high-k end because both the Q-field and the cosmological constant are spatially smooth on these scales. The clustering property of Q is reminiscent of the case of massive neutrinos in C+HDM models, which cannot cluster appreciably below the neutrino free-streaming scale but can cluster with the same amplitude as the cold dark matter on large scales. Analogous to the shape parameter Γ_ν that was introduced to model the neutrino streaming distances in C+HDM models (Ma 1996), we introduce a new shape parameter Γ_Q here to characterize the feature in Figure 1 in QCDM models. For a constant equation of state, w_Q, one can show that $V_{,QQ} = 6\pi G(1 - w_Q)(2\rho + p + w_Q\rho)$, where ρ and p are the total energy density and pressure. We approximate

$$k_Q = \Gamma_Q h = 2\sqrt{V_{,QQ}} = \frac{3H}{c}\sqrt{(1 - w_Q)[2 + 2w_Q - w_Q\Omega_m(a)]}, \tag{2}$$

and we use a simple ratio of polynomials to express the relative transfer function:

$$T_{Q\Lambda}(k,\ a) \equiv \frac{T_Q}{T_\Lambda} = \frac{\alpha + \alpha q^2}{1 + \alpha q^2}, \quad q = \frac{k}{\Gamma_Q h}, \tag{3}$$

where k is in units of Mpc^{-1} and α is a scale-independent but time-dependent coefficient that quantifies the relative amplitude of the matter density field δ_m on large and small length scales.

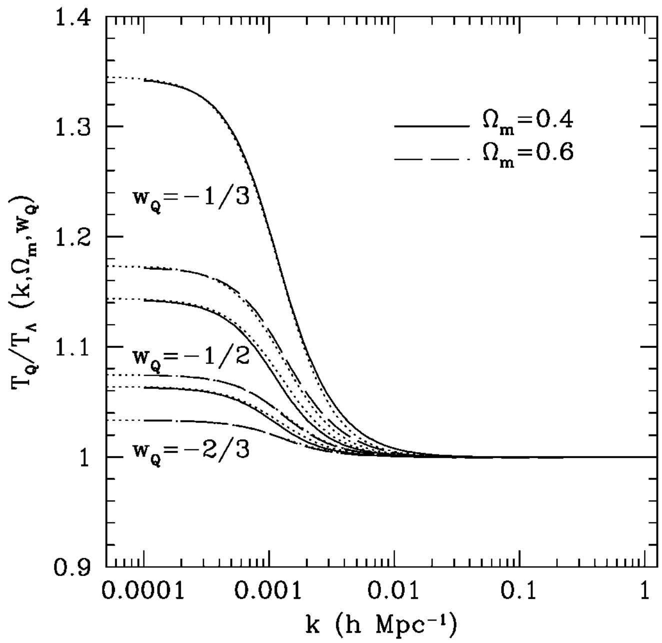

FIG. 1.—Ratio of the transfer functions, $T_{Q\Lambda} \equiv T_Q/T_\Lambda$, at the present day for six pairs of flat QCDM and ΛCDM models. The solid (for $\Omega_m = 0.4$) and dashed (for $\Omega_m = 0.6$) curves are computed from the Boltzmann integrations and illustrate the dependence on the matter density parameter Ω_m. For a given Ω_m, the three curves illustrate the dependence on the equation of state: $w_Q = -\frac{1}{3},\ -\frac{1}{2}$, and $-\frac{2}{3}$ from top down. The dotted curves show the analytic approximation given by eqs. (2)–(4). Note that $T_{Q\Lambda}$ deviates from unity only on very large length scale ($k \lesssim 0.01\ h\ \mathrm{Mpc}^{-1}$) above which the Q-field can cluster spatially.

We find α well approximated by

$$\alpha = (-w_Q)^s,$$
$$s = (0.012 - 0.036w_Q - 0.017/w_Q)[1 - \Omega_m(a)] + (0.098 + 0.029w_Q - 0.085/w_Q)\ln\Omega_m(a), \tag{4}$$

where the matter density parameter is $\Omega_m(a) = \Omega_m/[\Omega_m + (1 - \Omega_m)a^{-3w_Q}]$, which reaches the value Ω_m at the present-day $a = 1$. Figure 1 illustrates the close agreement (with errors $\lesssim$10%) between the approximations given by equations (2)–(4) and the exact results from numerical integrations of the Boltzmann equations.

Next we examine the linear growth suppression factor of the density field in equation (1). This function is well studied for ΛCDM models (Heath 1977; Lahav et al. 1991). An empirical fit is given by $g_\Lambda = 2.5\Omega_m(a)\{\Omega_m(a)^{4/7} - 1 + \Omega_m(a) + [1 + \Omega_m(a)/2][1 + (1 - \Omega_m(a))/70]\}^{-1}$ and is accurate to ~2% for $0.1 \le \Omega_m \le 1$ (Carroll, Press, & Turner 1992). This formula unfortunately cannot be generalized to QCDM models by simply replacing $(1 - \Omega_m) \to (1 - \Omega_m)a^{-3(1+w_Q)}$. Instead, we propose the following formula to approximate the ratio of the QCDM and ΛCDM growth factors:

$$g_{Q\Lambda} \equiv \frac{g_Q}{g_\Lambda} = (-w_Q)^t,$$
$$t = -(0.255 + 0.305w_Q + 0.0027/w_Q)[1 - \Omega_m(a)] - (0.366 + 0.266w_Q - 0.07/w_Q)\ln\Omega_m(a), \tag{5}$$

accurate to 2% for $0.2 \lesssim \Omega_m \leq 1$ and $-1 \leq w_Q \lesssim -0.2$. Figure 2 illustrates the dependence of $g_{Q\Lambda}$ on time, w_Q, and Ω_m. The growth is evidently slower in models with less negative w_Q for fixed Ω_m. This is because the energy density in the Q-field dominates over that in matter at an increasingly earlier time as w_Q is varied from -1 to 0; the growth of gravitational collapse therefore ceases earlier and results in a smaller value for $g_{Q\Lambda}$. (It is sometimes useful to study the instantaneous growth rate of δ_m, $f \equiv d \log \delta_m / d \log a$. See Wang & Steinhardt 1998 for a fitting formula for f.)

The remaining component in equation (1) to be specified is the normalization A_Q. It can be chosen by fixing the value of σ_8, the rms linear mass fluctuation within a top hat of radius 8 h^{-1} Mpc or by fixing to *COBE* results. For the latter we follow Bunn & White (1997). In the case that the temperature anisotropy is due to primordial adiabatic density perturbations with spectral index n, we write $A_Q = \delta_H^2 (c/H_0)^{n+3}/(4\pi)$, where

$$\delta_H = 2 \times 10^{-5} \alpha_0^{-1} (\Omega_m)^{c_1 + c_2 \ln \Omega_m} \times \exp\,[c_3(n-1) + c_4(n-1)^2],$$

$\alpha_0 = \alpha(a = 1)$ of equation (4), $c_1 = -0.789 \times |w_Q|^{0.0754 - 0.211 \ln |w_Q|}$, $c_2 = -0.118 - 0.0727 w_Q$, $c_3 = -1.037$, and $c_4 = -0.138$, for $-1 \lesssim w_Q \lesssim -0.2$. In the case of tensor perturbations, the primordial amplitudes follow the inflationary relation $A_T = 8(1-n)A_S$. The ratio of $\ell = 10$ multipole moments, $r_{10} = C_{10}^T / C_{10}^S$, is given by

$$r_{10} \approx 0.48(1-n)[1 + 0.1(1-n)(8 + 7w_Q)\Omega_Q^2 + 3] \times (1 - \Omega_Q/x)^{g_{10}(\Omega_Q/x)},$$

where $g_{10}(y) = 0.18 + 0.84y^2$ and $x = 0.75[1 - 0.66w_Q + 1.66w_Q^2 - 0.5(1 + w_Q)^5]$. One can rescale A_Q by $A_Q \rightarrow A_Q/(1 + r_{10})$ to accommodate the effect of tensors on the normalization.

3. NONLINEAR MASS POWER SPECTRUM

In this section we examine if the simple linear to nonlinear mapping technique initiated by Hamilton et al. (1991) can be extended to QCDM models. The basic approach is to search for a simple expression for the function $\Delta_{\rm nl}(k) = f[\Delta_l(k_0)]$ that relates the linear and nonlinear density variance $\Delta(k) \equiv 4\pi k^3 P(k)$. Note that $\Delta_{\rm nl}$ and Δ_l are evaluated at different wavenumbers, where $k_0 = k(1 + \Delta_{\rm nl})^{-1/3}$ corresponds to the precollapsed scale of k. The strategy is to combine analytical clustering properties in asymptotic regimes with fits to numerical simulation results. This recipe has been successfully developed for scale-free models with a power law $P(k)$ (Hamilton et al. 1991; Jain et al. 1995), flat CDM and ΛCDM models (Jain et al. 1995; Peacock & Dodds 1996, hereafter PD96; Ma 1998, hereafter Ma98), and flat C+HDM models with massive neutrinos (Ma98).

We investigate if the PD96 and Ma98 formulas proposed for ΛCDM models can be easily extended to QCDM models. These two formulae incorporate the time dependence of the mapping in different ways, but they share the feature that the dependence on parameters Ω_m and Ω_Λ enters only through the linear growth factor g. In order to test the application of this method to QCDM models, we have performed N-body simulations for three values of w_Q: $-\frac{2}{3}$, $-\frac{1}{2}$, and $-\frac{1}{3}$, each with several different realizations. These three values should be sufficient since extensive tests of $w_Q = -1$ (i.e., ΛCDM models) have already been carried out in PD96 and Ma98. We restrict our attention to $w_Q < -\frac{1}{3}$ and cosmological parameter ranges that are in concordance with observations (Wang & Steinhardt 1998; Wang et al. 1999). Specifically, $(\Omega_m, \Omega_Q, \Omega_b, h) = (0.4, 0.6, 0.047, 0.65)$ for the $w_Q = -\frac{2}{3}$ and $-\frac{1}{2}$ models and $(\Omega_m, \Omega_Q, \Omega_b, h) = (0.45, 0.55, 0.047, 0.65)$ for the $w_Q = -\frac{1}{3}$ model. The N-body code used is a parallel version of the particle-particle particle-mesh algorithm (Bertschinger & Gelb 1991; Ferrell & Bertschinger 1994). Each simulation uses 128^3 particles in a box of comoving volume 100^3 Mpc^3. The Plummer softening length is 50 kpc comoving, which allows us to compute the nonlinear power spectrum in highly clustered regions with $k \lesssim 10$ h Mpc^{-1} and $\Delta_{\rm nl} \lesssim 1000$. Since the Q-field clusters only on scales much above the box size, the presence of the quintessence only affects the initial conditions and the evolution of the scale factor a.

Figure 3 compares the linear power spectrum and the fully evolved spectrum from both the N-body runs and the approximations of PD96 and Ma98. Five redshifts are shown for each of three QCDM models. Overall, we find that the PD96 formula works well at $z = 0$ when the factor g in their formula is set to $g = g_Q$, which is the appropriate growth factor for the density field for QCDM models [eq. (5)]. At earlier times, however, the PD96 formula *underestimates* $\Delta_{\rm nl}$ at $k \gtrsim 1$ h Mpc^{-1} in the $w_Q = -\frac{2}{3}$ and $-\frac{1}{2}$ models by up to 30%. We have attempted less physically motived combinations of growth factors (e.g., $g = \alpha\, g_Q$ and g_Λ) but did not find a way to make PD96 fit.

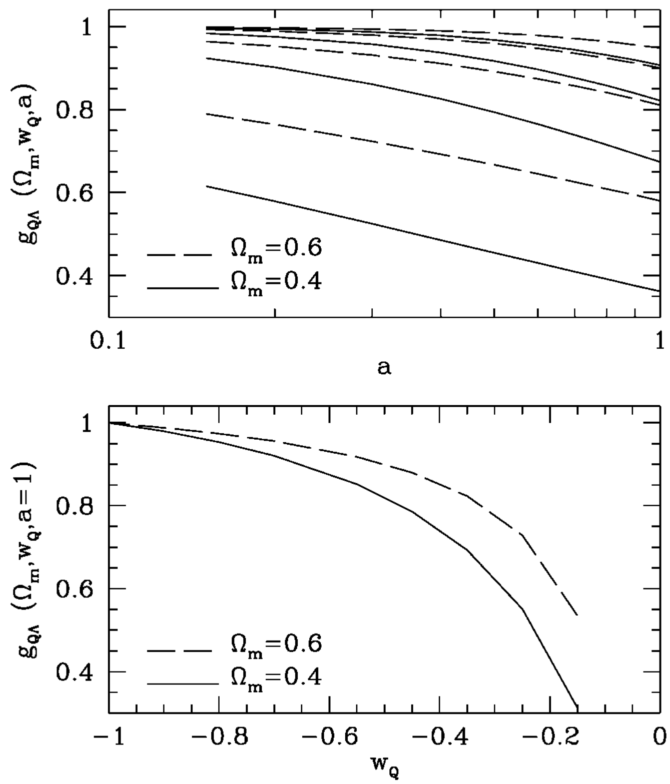

FIG. 2.—Ratio of the growth suppression factors, $g_{Q\Lambda} \equiv g_Q/g_\Lambda$, as a function of the scale factor a (*top*) and the equation of state w_Q (*bottom*; at $a = 1$) for various pairs of flat QCDM and ΛCDM models. The solid and dashed curves in both panels are for $\Omega_m = 0.4$ and 0.6, respectively. In the top panel, each set of curves corresponds to $w_Q = -\frac{2}{3}$, $-\frac{1}{2}$, $-\frac{1}{3}$, and $-\frac{1}{6}$ from top down.

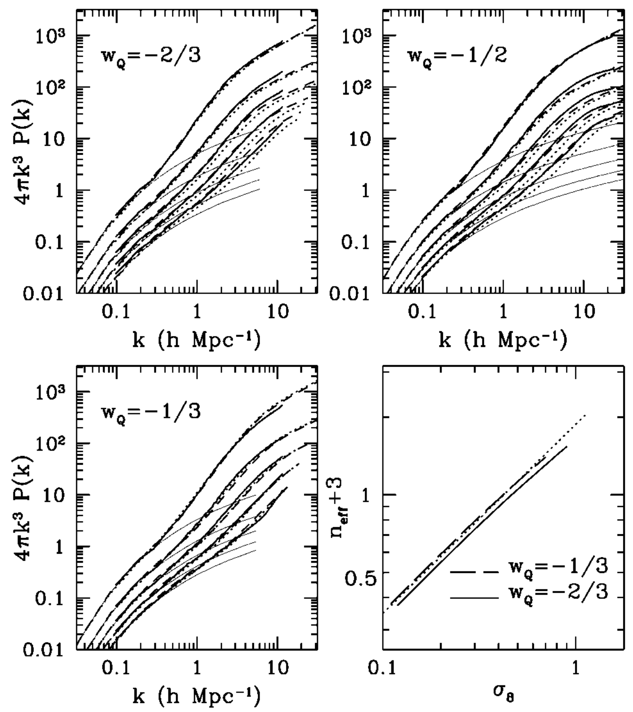

FIG. 3.—*Top row and lower left*: The linear and fully evolved nonlinear power spectra for the matter density field in QCDM models with different equations of state w_Q (see text for other model parameters). In each panel, five redshifts, $z = 0$, 1, 2, 3, and 4, are shown (*top down*). The curves are computed from *N*-body simulations directly (*thick solid line*), nonlinear approximation by Ma98 (*dashed line*; eq. [6] of this Letter) and PD96 (*dotted line*), and linear theory (*thin solid line*). *Lower right*: The effective spectral index, $n_{\rm eff}$, as a function of σ_8 for two QCDM models. The dotted line represents a power law and demonstrates that $d\ln(n_{\rm eff}+3)/d\ln\sigma_8 \propto \beta$ is an excellent approximation.

We find that the Ma98 formula,

$$\frac{\Delta_{\rm nl}(k)}{\Delta_l(k_0)} = G\left(\frac{\Delta_l}{g_0^{3/2}\sigma_8^{\beta}}\right),$$

$$G(x) = [1 + \ln(1+0.5x)]\,\frac{1+0.02x^4+c_1x^8/g^3}{1+c_2x^{7.5}}, \quad (6)$$

can be easily extended to QCDM models. Specifically, we propose to keep $c_1 = 1.08\times10^{-4}$ and $c_2 = 2.10\times10^{-5}$ used for ΛCDM in Ma98, but adopt $g = g_Q$, which is the appropriate QCDM growth factor (eq. [5]), and $g_0 = |w_Q|^{1.3|w_Q|-0.76} g_{Q,0}$, where $g_{Q,0} \equiv g_Q(a=1)$. As described in Ma98, the parameter β in equation (6) is introduced to approximate the power-law dependence of the effective spectral index $n_{\rm eff}+3$ in previous work on σ_8: $d\ln(n_{\rm eff}+3)/d\ln\sigma_8 \propto \beta$. We find $\beta = 0.83$ an excellent approximation for all three QCDM models that we tested (see the lower right-hand panel of Fig. 3). Other panels of Figure 3 illustrate the close agreement (rms errors of ∼10%) between *N*-body results and equation (6).

4. SUMMARY

We have presented simple formulas to approximate both the linear and nonlinear power spectra for matter density perturbations in viable quintessence cosmological models with an equation of state $-1 \le w_Q \lesssim -\frac{1}{3}$. Equations (2), (3), and (4) together specify the ratio of the linear transfer functions T_Q and T_Λ for the matter density field for a given pair of QCDM and ΛCDM models with the same cosmological parameters. Equation (5) specifies the ratio of the linear growth suppression factors g_Q and g_Λ in QCDM and ΛCDM models. Equation (6) approximates the nonlinear mass power spectrum.

A key difference between gravitational clustering in QCDM and ΛCDM models is that Λ is spatially smooth on all length scales, whereas the Q-field can cluster above a certain length scale. We characterize this length scale by the shape parameter Γ_Q of equation (2), which is derived from the Compton wavelength of the Q-field. The QCDM matter power spectrum therefore changes shape at two characteristic scales: Γ_Q, and the familiar $\Gamma \propto \Omega_m h$ that corresponds to the crossover from radiation- to matter-dominated era. For the QCDM models studied in this Letter (i.e., constant w_Q), the Compton wavelength of the Q-field is very large: $k_Q \sim 0.001$ to 0.01 h Mpc^{-1} (see Fig. 1). On scales of galaxy clusters and below, therefore, the linear QCDM power spectrum has *identical* shape as in the corresponding ΛCDM model and differs only in the overall amplitude by a factor of $(A_Q/A_\Lambda)(g_{Q\Lambda}/g_{Q\Lambda,0})^2$. This realization should simplify comparisons between QCDM and ΛCDM models.

For the fully evolved nonlinear power spectrum, we find that PD96 works well at $z = 0$ but underestimates its amplitude by up to ∼30% at earlier times. The formula of Ma98, on the other hand, can be easily extended to approximate the QCDM nonlinear $P(k)$ (with errors ≲10%) for $w_Q \lesssim -\frac{1}{3}$ and redshift up to $z \approx 4$. Equation (6) summarizes this result.

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