The power spectrum

The power spectrum of density fluctuations is defined as

$$P(k) \equiv \langle |\delta_k|^2 \rangle$$

where $\delta_{\mathbf{k}}$ is the Fourier transform of the density perturbation field. Angle brackets denote an ensemble average. Notice that in the linear regime we have

$$\delta_k \propto D_+(z) \quad \Rightarrow \quad P(k) \propto D_+(z)^2$$

In general we could have P be a function of the amplitude and direction of **k**, but in an isotropic universe P must be a function of the amplitude k only.

The *dimensionless power spectrum* is defined as

$$\Delta^2(k) \equiv \frac{V}{(2\pi)^3} 4\pi k^3 P(k)$$

where V is the normalization volume (arbitrary).

Δ^2 can also be thought of as the mass variance per ln k. When $\Delta^2(k) = 1$, we have order-unity fluctuations in the modes in a logarithmic wavenumber interval about k.

The power spectrum – 2

For a Gaussian random field (such as those produced in inflationary models), the power in perturbation modes with comoving wavenumber k is distributed according to the *Rayleigh distribution*:

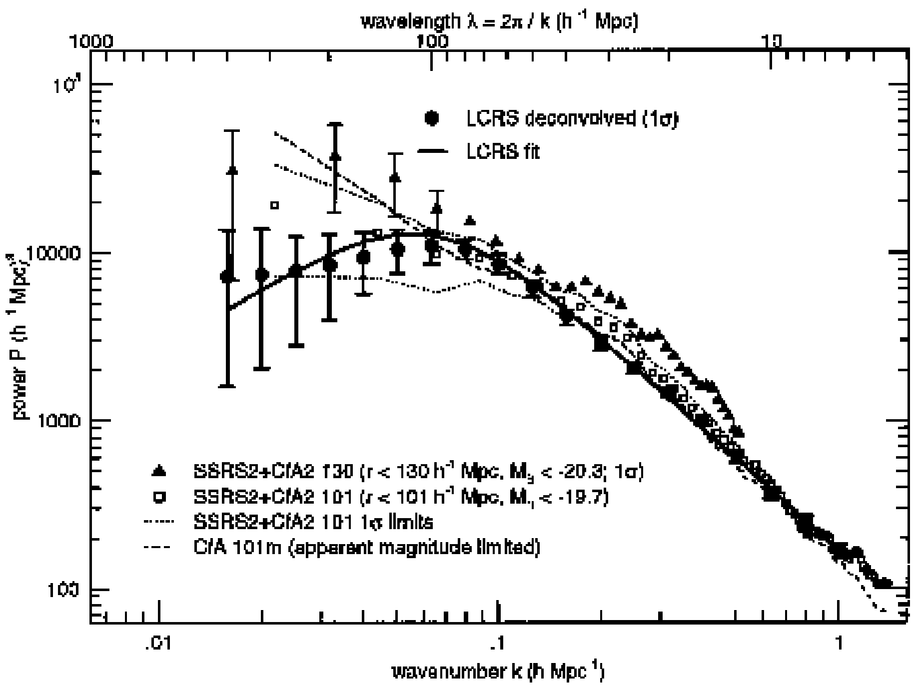
$$\text{Probability that } |\delta_k|^2 > X \text{ is } \exp\left[-X^2/P(k)\right]$$

The complex phase of δ_k is uniformly distributed in $[0, 2\pi)$.

A *realization* of a given power spectrum in a finite volume will display fluctuations in the actual power about $P(k)$ due to the finite number of modes (& hence samples) with a given wavenumber k.

This is called *cosmic variance*.

The largest variance comes at small k, where we have the smallest number of samples.



The power spectrum – 3

For Gaussian random fields the power spectrum tells us everything there is to know about the density perturbation field.

In models with Gaussian random perturbations we write the linear power spectrum at late times as the product of a *primordial power spectrum* and a *transfer function* $T(k)$ due to the growth of dark matter fluctuations:

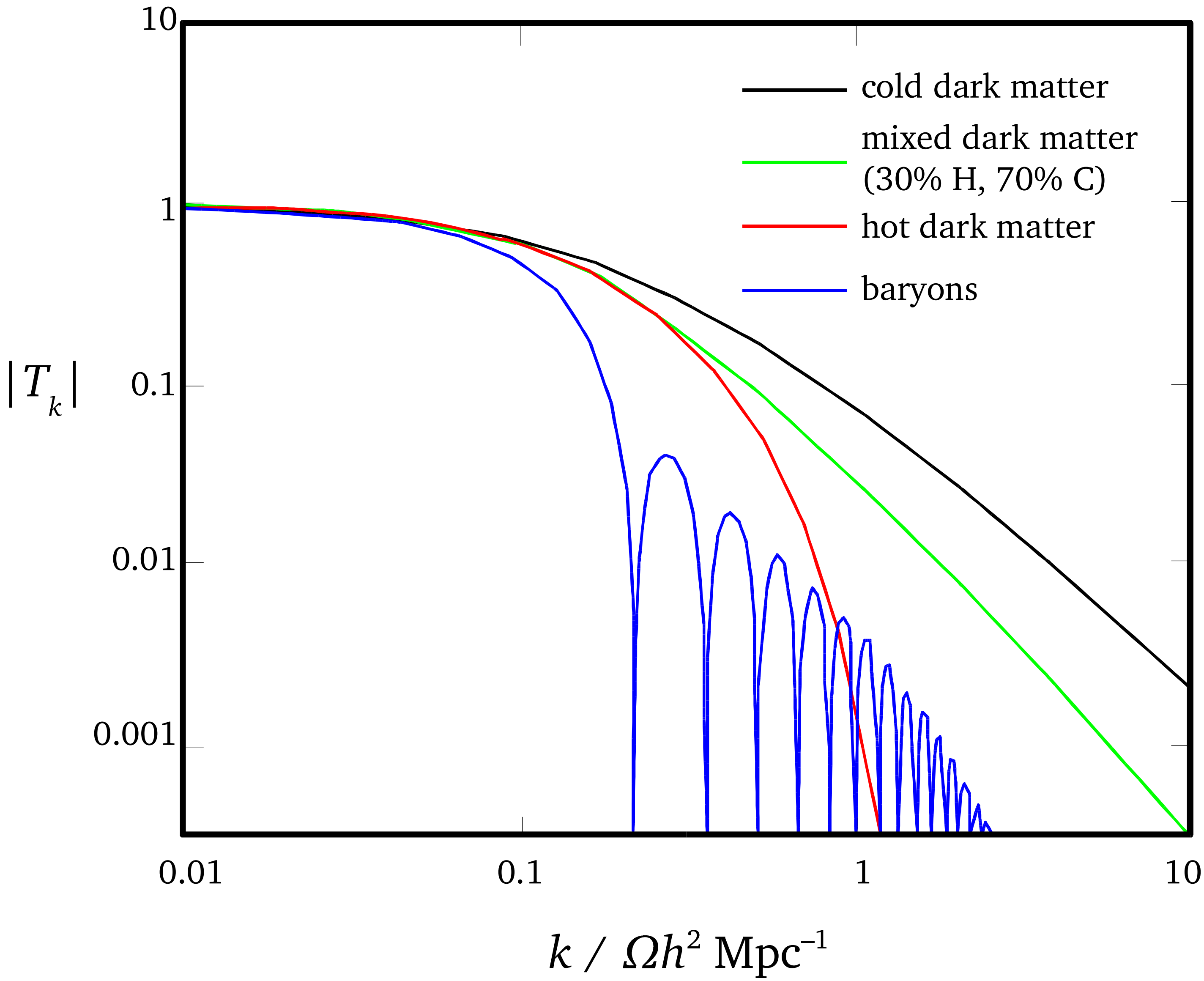
$$P(k,z) = AD_+(z)^2 k^n T(k)^2, \quad A = \text{constant}$$

The primordial power spectrum is characterized by a *spectral index*, n, which may in principle vary with k. In the absence of other information the *Harrison-Zel'dovich spectrum* ($n = 1$) is often used because it corresponds to a scale-invariant dimensionless potential fluctuation.

The transfer function depends on the type of dark matter present. It breaks the scale invariance of the primordial spectrum by introducing new length scales, e.g.,

- the horizon size at decoupling of the dark matter
- the horizon size at matter-radiation equality
- the horizon size at the time the dark matter becomes nonrelativistic

Transfer functions in different models (adiabatic)



The mass variance

The *mass variance* is defined from the power spectrum by filtering on a given length scale R:

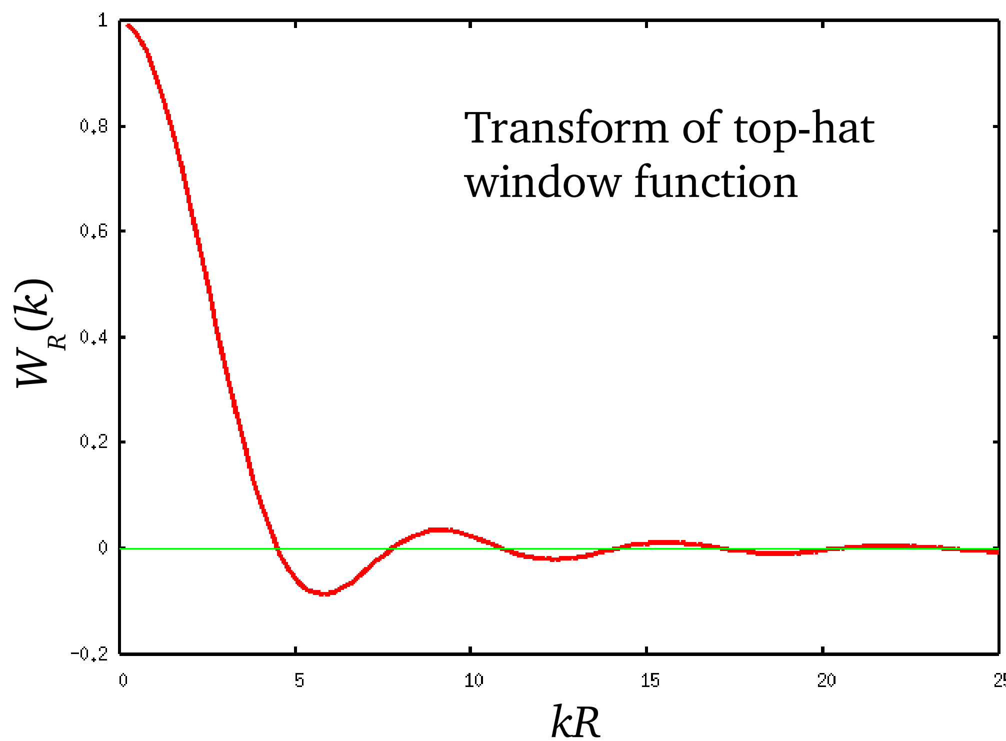
$$\sigma^2(R) \equiv 4\pi \int_0^\infty P(k) |W_R(k)|^2 k^2 \, dk$$

Usually we use the *top-hat filter*:

$$W_R(r) = \begin{cases} 3/(4\pi R^3) & r \leq R \\ 0 & r > R \end{cases} \quad \Rightarrow \quad W_R(k) = \frac{3}{(kR)^3}\left[\sin kR - kR \cos kR\right]$$

The mass variance can be defined as a function of mass scale M via

$$M = \frac{4\pi}{3} R^3 \bar{\rho}$$



The mass variance – 2

We can use the mass variance on a fixed scale at the present day to normalize the power spectrum: if

$$P(k, z) = A D_+(z)^2 k^n T(k)^2$$

and

$$D_+(z=0) = 1$$

then

$$\sigma_R^2 = 4\pi A \int_0^\infty k^{2+n} T^2(k) |W_R(k)|^2 \, dk$$

$$A = \frac{\sigma_R^2}{4\pi \int_0^\infty k^{2+n} T^2(k) |W_R(k)|^2 \, dk}$$

Typically the comoving normalization scale used is $R = 8\ h^{-1}$ Mpc, the approximate scale on which structure is becoming nonlinear today:

$$\sigma_8 \equiv \sigma(8\ h^{-1}\,\mathrm{Mpc}) \approx 0.8 - 0.9$$

from galaxy cluster surveys.

Zel'dovich pancakes

Zel'dovich (1970) showed that nonlinear collapse in a random perturbation field is more likely to be *plane-parallel* than spherical.

Consider the problem of collapse from a Lagrangian viewpoint: the proper coordinate **r** of a particle with initial comoving wavenumber **q** is

$$r = a(t)\left[\mathbf{q} + D_{+}(t)\,\mathbf{p}(\mathbf{q})\right]$$

The first term describes the cosmological expansion. The second term describes the peculiar motion produced by the density perturbations.

- $D_{+}(t)$ is the growth factor for linear perturbations – in a universe with $\Omega = 1$ we have $D_{+}(t) = a(t)$.

- $\mathbf{p}(\mathbf{q})$ contains information about the initial perturbation field.

Zel'dovich pancakes – 2

The deformation (strain) tensor is

$$D_{ij} = \frac{\partial r_i}{\partial q_j} = a(t)\delta_{ij} + a(t)D_+(t)\frac{\partial p_i}{\partial q_j}$$

The principal axes of the deformation are determined by $\partial p/\partial q$; because the flow is irrotational, $\boldsymbol{D}$ is symmetric in this coordinate system:

$$\boldsymbol{D} = \begin{bmatrix} a - \alpha a D_+ & 0 & 0 \\ 0 & a - \beta a D_+ & 0 \\ 0 & 0 & a - \gamma a D_+(t) \end{bmatrix}$$

From conservation of mass we can obtain the density ρ:

$$\rho a^3 (1 - \alpha D_+)(1 - \beta D_+)(1 - \gamma D_+) = \bar{\rho} a^3$$

Notice that the density becomes infinite when any one of the terms in parentheses becomes zero. For random perturbations, the chance that any two of (α, β, γ) are the same, or that all are the same, is vanishing – hence collapse should occur preferentially along one direction (the direction depends on the local pert. field).

Zel'dovich pancakes – 3

The *Zel'dovich approximation* can be used both to set up cosmological simulations and as a single-mode test problem:

$$\mathbf{x}(t) = \mathbf{q} + D_+(t)\,\mathbf{p}(\mathbf{q})$$
$$\mathbf{v}(t) = \dot{D}_+(t)\,\mathbf{p}(\mathbf{q})$$

Notice that, as in Eulerian perturbation theory, we have

$$\delta \equiv \frac{\rho}{\bar{\rho}} - 1 \approx -D_+(\alpha + \beta + \gamma) = -D_+ \nabla \cdot \mathbf{p} \quad \Rightarrow \quad \nabla \cdot \mathbf{v} = -\dot{\delta}$$

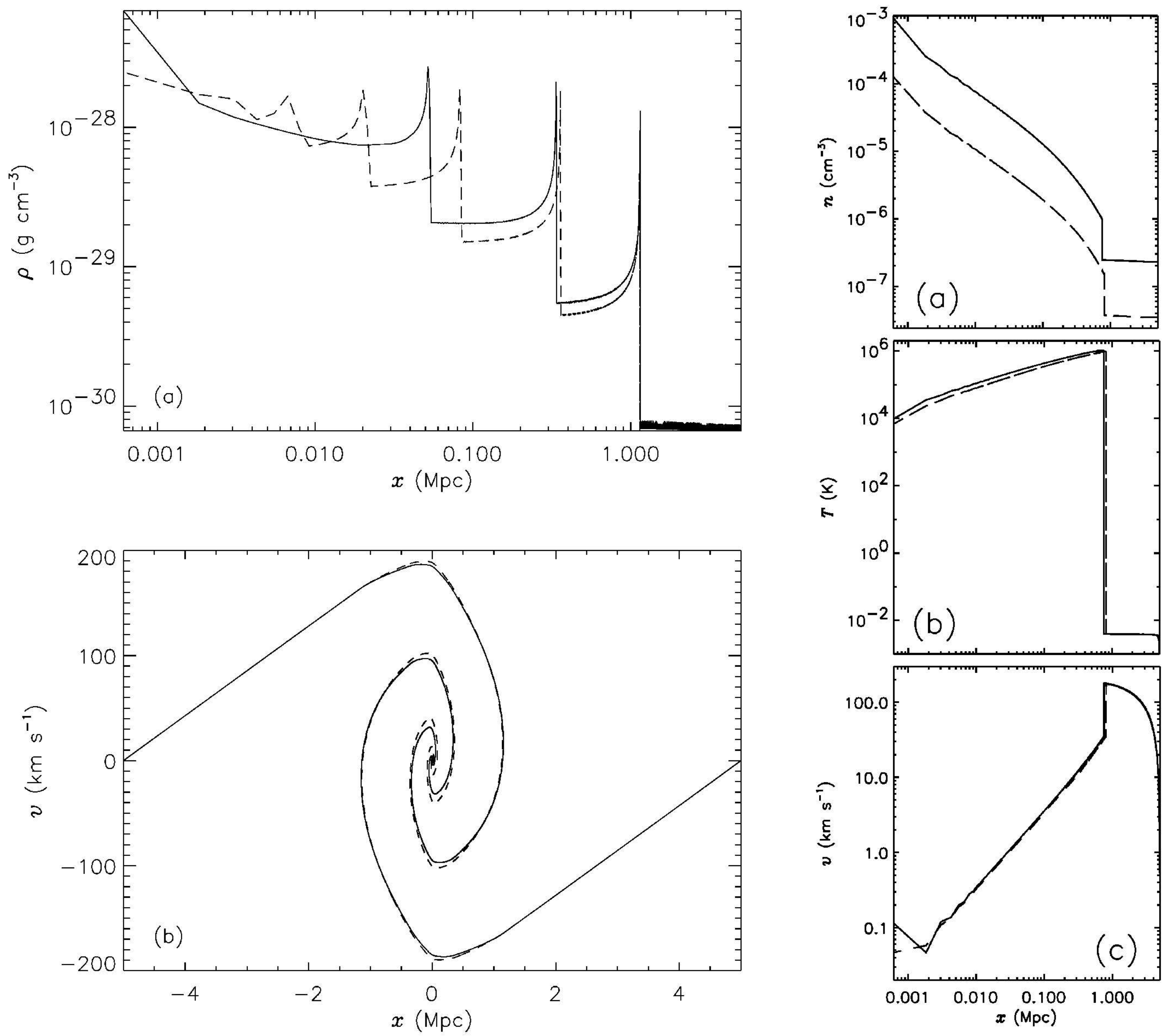
However, the Zel'dovich approximation works well even up until $\delta \sim 1$, whereas Eulerian linear perturbation theory begins to break down much earlier.

For a 1D single-mode test problem we can use

$$p(q) = \epsilon \sin(\kappa q)$$

where $\epsilon \ll 1$ and κ is the perturbation wavenumber. For baryons (gas) we can also set up a pressure profile assuming adiabatic compression from a constant-temperature state. The detailed nonlinear evolution of mixed dark matter/gas pancakes has been investigated by Anninos & Norman (1994).

Zel'dovich pancakes – 4



FLASH solution to the Zel'dovich pancake problem (Zel'dovich 1970; Anninos & Norman 1994) in a flat Universe ($\Omega_0 = 1$) with Hubble constant $H_0 = 50$ km s^{-1} Mpc^{-1} and comoving wavelength $\lambda = 10$ Mpc at redshift $z = 0$. **Left** Dark matter density profile (top) and phase plot (bottom). Solid curves show dark-matter-only case; dashed curves show case with 10% gas fraction. **Right** Gas number density, temperature, and velocity profiles. Solid curves show gas-only case; dashed curves show case with 10% gas fraction. A uniform one-dimensional mesh with 8,192 zones was used for the gasdynamics and potential solver, and 65,536 particles were used to represent the dark matter.

Initializing cosmological simulations (uniform meshes)

1. Compute the Fourier transform of the density fluctuation field, $\delta_{\mathbf{k}} = |\delta_{\mathbf{k}}| \exp(i\theta_{\mathbf{k}})$:

   For each k-space zone pqr, $|\delta_{pqr}| = D_+(z)\sqrt{P(k_{pqr}, z=0)\eta}$
   η an exponential deviate.

   phase $\theta_{pqr} = 2\pi\zeta$ ζ a uniform deviate in [0,1).

   Note 1: must have $\delta_{N-p,N-q,N-r} = \delta_{pqr}$, $\theta_{N-p,N-q,N-r} = -\theta_{pqr}$ since $\delta(\mathbf{x})$ is real-valued.

   Note 2: usually choose initial redshift z so that $\max[\delta(\mathbf{x})] = 1$.

2. Inverse Fourier transform to get the real-space density fluctuation $\delta_{ijk} = \delta(\mathbf{x}_{ijk})$.

3. To get the velocity field, use $\nabla \cdot \mathbf{v} = -\dot{\delta}$ and the fact that the velocity is potential:

$$\mathbf{v}_{pqr} = \frac{i\,\mathbf{k}_{pqr}}{k_{pqr}^2} \frac{\dot{D}_+}{D_+} \delta_{pqr}$$

   then inverse Fourier transform to get $\mathbf{v}_{ijk} = \mathbf{v}(\mathbf{x}_{ijk})$.

Initializing cosmological simulations (particles)

1. Take unperturbed positions **q** to lie on a grid: $\mathbf{q}_{ijk} = (i\,\Delta x\,,\, j\,\Delta y\,,\, k\,\Delta z)$

2. Compute the Fourier transform of the velocity potential ψ and velocity **v**:

$$\mathbf{v} = \nabla\psi \quad \Rightarrow \quad \mathbf{v}_{pqr} = i\,\mathbf{k}_{pqr}\,\psi_{pqr}$$

$$\nabla\cdot\mathbf{v} = -\dot{\delta} \quad \Rightarrow \quad \nabla^2\psi = -\dot{\delta}$$

$$\psi_{pqr} = \frac{\dot{\delta}_{pqr}}{k^2_{pqr}}$$

$$\mathbf{v}_{pqr} = \frac{i\,\mathbf{k}_{pqr}}{k^2_{pqr}}\dot{\delta}_{pqr}$$

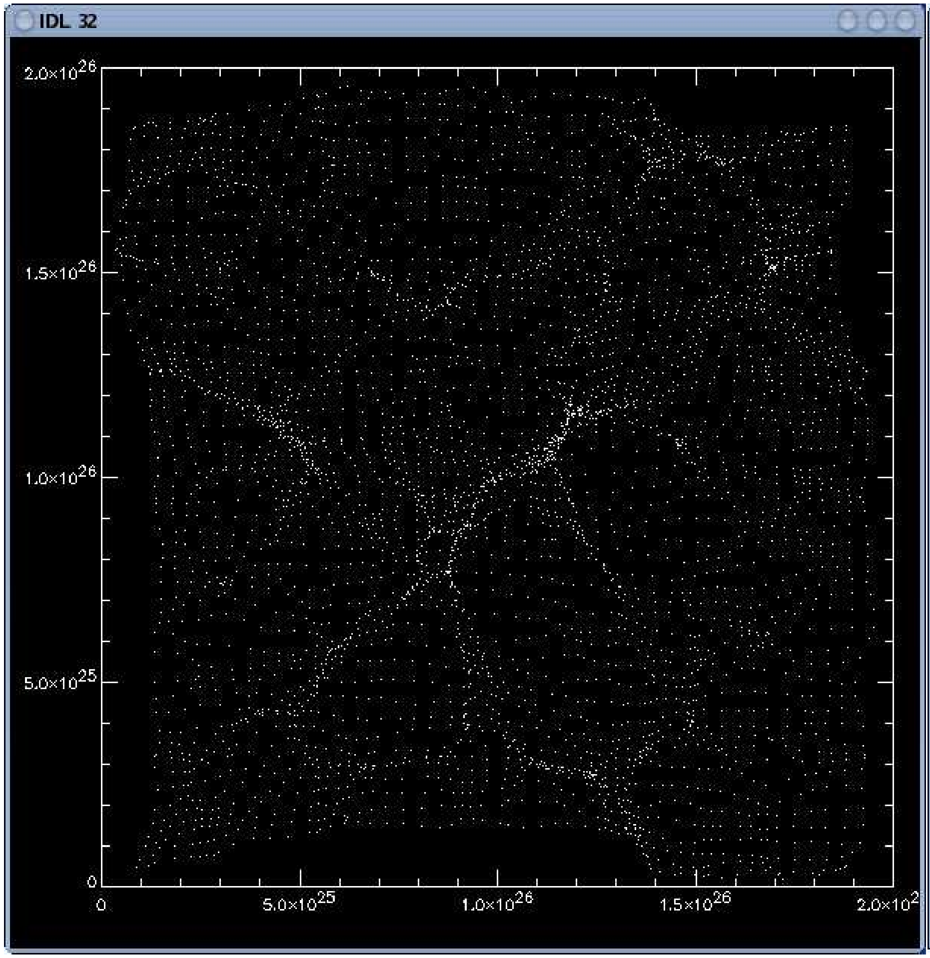
The $\dot{\delta}_{pqr}$ are computed as for grid-based initialization.

3. Inverse Fourier transform to get the particle velocities. The displaced particle positions are then

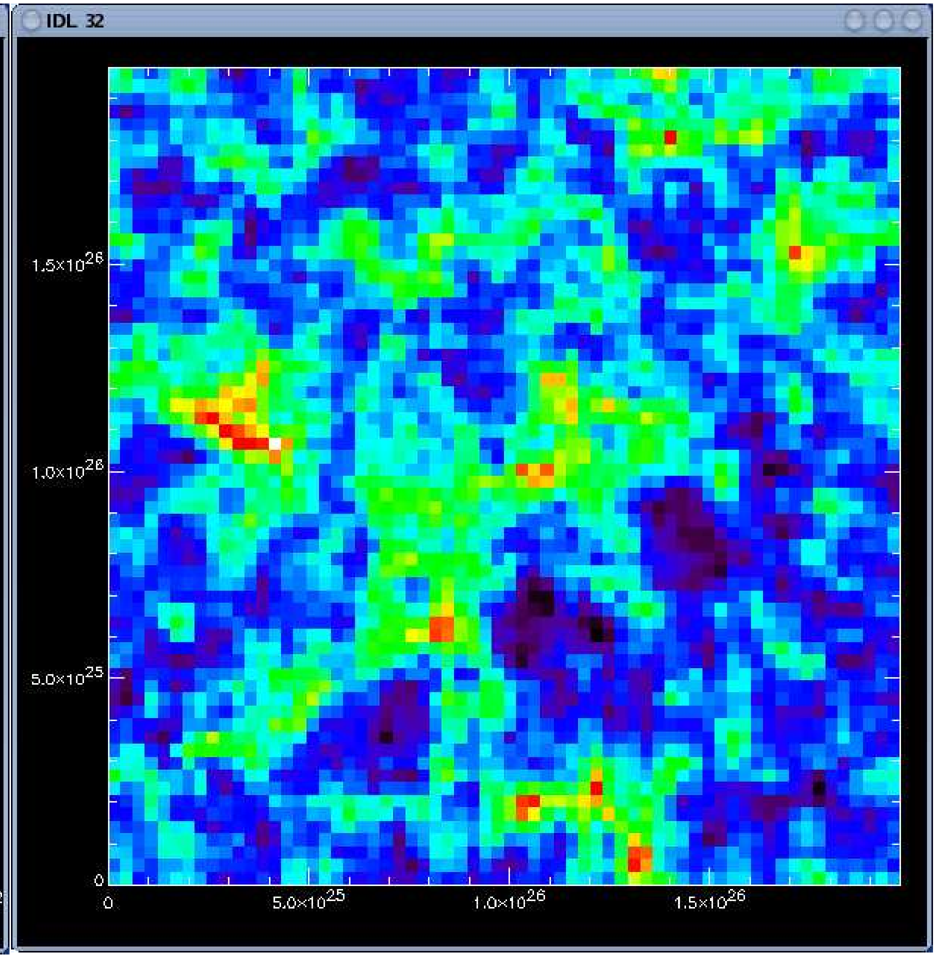
$$\mathbf{x}_{ijk} = \mathbf{q}_{ijk} + \frac{D_+}{\dot{D}_+}\mathbf{v}_{ijk}$$

Zel'dovich approximation example

Dark matter particle positions



Mesh gas overdensities



(displacements multiplied by 7)